Symmetric periodic orbits near a heteroclinic loop in $\mathbb{R}^3$ formed by two singular points, a semistable periodic orbit and their invariant manifolds

Montserrat Corbera [a,1] Jaume Llibre [b,2]
Marco Antonio Teixeira [c,3]

[a] *Departament de Tecnologies Digitals i de la Informació, Universitat de Vic Laura 13, 08500 Vic, Barcelona, Catalonia, Spain.*

[b] *Departament de Matemàtiques, Universitat Autònoma de Barcelona 08193 – Bellaterra, Barcelona, Catalonia Spain.*

[c] *Departamento de matemática, Universidade Estadual de Campinas, Caixa Postal 6065, 13083–970 Campinas SP, Brazil.*

Abstract

In this paper we consider $\mathcal{C}^1$ vector fields X in $\mathbb{R}^3$ having a "generalized heteroclinic loop" $\mathcal{L}$ which is topologically homeomorphic to the union of a 2–dimensional sphere $\mathbb{S}^2$ and a diameter Γ connecting the north with the south pole. The north pole is an attractor on $\mathbb{S}^2$ and a repeller on Γ. The equator of the sphere is a periodic orbit unstable in the north hemisphere and stable in the south one. The full space is topologically homeomorphic to the closed ball having as boundary the sphere $\mathbb{S}^2$. We also assume that the flow of X is invariant under a topological straight line symmetry on the equator plane of the ball. For each $n \in \mathbb{N}$, by means of a convenient Poincaré map, we prove the existence of infinitely many symmetric periodic orbits of X near $\mathcal{L}$ that gives n turns around $\mathcal{L}$ in a period. We also exhibit a class of polynomial vector fields of degree 4 in $\mathbb{R}^3$ satisfying this dynamics.

Key words: Heteroclinic loop, symmetric periodic orbits, polynomial vector fields.
1991 MSC: 37C27, 37C29, 37C80

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1 Introduction

In this paper we shall analyze the dynamics of $\mathcal{C}^1$ vector fields X in $\mathbb{R}^3$ having a "generalized heteroclinic loop" $\mathcal{L}$. It is well known that the dynamics of a vector field in a neighborhood of a homoclinic or a heteroclinic loop may be very rich and complex. So, from a dynamical point of view, the study of the behavior of a vector a field near these objects is very interesting.

Usually the heteroclinic loops that have been studied in the literature come from heteroclinic loops having a transversal intersection of a stable invariant manifold and the unstable one along their orbits different from singular points, see for instance (7; 9; 15). These heteroclinic loops generically have the Bernoulli shift as a subsystem and consequently the vector field possesses infinitely many periodic orbits near the heteroclinic loop. The mechanism that generates the infinitely many periodic orbits near these heteroclinic loops is the transversality, whereas for the generalized heteroclinic loops studied in this paper the mechanism is the symmetry. The origin of the transversality and of the Bernoulli shift can be found in (15), but one of the best applications of this dynamics can be found in (9) where the author uses a heteroclinic transversal loop in order to study the dynamics near the parabolic orbits of the Sitnikov problem.

In this paper our generalized heteroclinic loop has no transversal intersection between the stable and unstable invariant manifolds, these invariant manifolds coincide. So it does not posses the Bernoulli shift as a subsystem, see for more details (9; 10). Nevertheless it still conserves infinitely many periodic orbits near it. There are very few articles in the literature studying heteroclinic loops that have no transversal intersection between their stable and unstable invariant manifolds and that these coincide, see for instance (4; 5). Moreover, usually the invariant manifolds that appear associated to the heteroclinic loops studied in the literature have the same dimension. In our heteroclinic loop these objects have different dimension. From the point of view of having invariant manifolds with different dimension our heteroclinic loop can recall to the homoclinic loops of Silnikov type (see for example (1; 8; 11; 12)), but the main difference with these kind of loops is that their invariant manifolds cannot coincide as in our heteroclinic loop.

Email addresses: `montserrat.corbera@uvic.cat` (Montserrat Corbera), `jllibre@mat.uab.cat` (Jaume Llibre), `teixeira@ime.unicamp.br` (Marco Antonio Teixeira).

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Of course there are many other kinds of phenomena related with homoclinic and heteroclinic loops, as for instance the ones related with blue sky catastrophes (see for instance (13; 14)), or the ones related with homoclinic snaking (for example see (2; 3; 6; 16)), and several others. But all these other phenomena are different to our generalized heteroclinic loops.

Modulo a diffeomorphism we assume that $\mathcal{L}$ is the union of the sphere $\mathbb{S}^2$ of radius 1 centered at the origin of $\mathbb{R}^3$, and the open diameter Γ along the z–axis.

Let $X = (f(x, y, z), g(x, y, z), h(x, y, z))$ be a $\mathcal{C}^1$ vector field defined on the closed ball $\mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leqslant 1\}$. We denote $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, by $\mathbb{S}^+ = \mathbb{S}^2 \cap \{z > 0\}$ and $\mathbb{S}^- = \mathbb{S}^2 \cap \{z < 0\}$. Assume that X satisfies the following conditions:

- (C_1) The vector field X has two hyperbolic singular points on $\mathbb{S}^2$, $e^+ = \{(0, 0, 1)\}$ and $e^- = \{(0, 0, -1)\}$, which are nodes on $\mathbb{S}^2$.
- (C_2) The equator of the sphere $\Lambda = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ z = 0\}$ is a semistable periodic orbit such that $W^s(\Lambda) = W^u(e^-) = \mathbb{S}^- \setminus \{e^-\}$ and $W^u(\Lambda) = W^s(e^+) = \mathbb{S}^+ \setminus \{e^+\}$.
- (C_3) $W^u(e^+) = W^s(e^-) = \Gamma = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0,\ -1 < z < 1\}$.
- (C_4) In a small neighborhood U of Λ in $\mathbb{D}^3$, the flow of X satisfies that $\dot{z} > 0$ in $U \setminus \Lambda$.
- (C_5) The flow on $\mathbb{D}^3$ is invariant under the time–reversibility symmetry $s : (x, y, z, t) \longrightarrow (-x, y, -z, -t)$. That is, it is symmetric with respect to the y–axis, denoted by the straight line L in what follows.

Under these assumptions the vector field X possesses a generalized heteroclinic loop $\mathcal{L}$ formed by three invariant objects (the singular points e^+ and e^- and the semistable periodic orbit Λ) and their invariant manifolds.

It is well known that a periodic orbit Γ on a 2–dimensional sphere $\mathbb{S}^2$ can be either stable, unstable or semistable. In the first case Γ is a local attractor on $\mathbb{S}^2$, in the second case is a local repeller on $\mathbb{S}^2$, and in the last case in one side of $\mathbb{S}^2$ is a local attractor and in the other side is a local repeller.

In this paper, by means of a convenient Poincaré map π, we will prove that the existence of this generalized heteroclinic loop together with the symmetry of the vector field X guarantees the existence of infinitely many symmetric periodic orbits near $\mathcal{L}$. The key point for proving this result is that the image by our Poincaré map of the segment on the line of symmetry $\gamma = \{(0, y, 0) : 0 < y < \varepsilon\}$, where ε is sufficiently small, is a spiral on the plane $z = 0$ that approaches Λ spiraling infinitely many times. The intersection points of this spiral with the line of symmetry L give the infinitely many symmetric periodic orbits. In particular, in Section 2, we will prove the following result.

Theorem 1 *Assume that the vector field X is defined on the closed ball $\mathbb{D}^3$ satisfying the conditions (C_1)–(C_5). For each $n \in \mathbb{N}$ the vector field X has infinitely many symmetric periodic orbits near the heteroclinic loop $\mathcal{L}$ that cross exactly $2n$ times the plane $z = 0$ during a period.*

We remark that in this paper when we say that an orbit is periodic of period T this always means that T is the minimal period of the orbit.

We note that all the periodic orbits with the same number of crossings with $z = 0$ accumulate to the heteroclinic loop as we shall see in the proof of Theorem 1. Consequently their period tends to infinity when the periodic orbit tends to the loop.

In (5), motivated by models of Celestial Mechanics, an analogue of Theorem 1 for a class of vector fields in $\mathbb{D}^3$ having a generalized heteroclinic loop formed by two singular points e^+ and e^- that are foci on $\mathbb{S}^2$, and their invariant manifolds satisfying $W^u(e^-) = W^s(e^+) = \mathbb{S}^2 \setminus \{e^+, e^-\}$ and $W^u(e^+) = W^s(e^-) = \Gamma$ is proven. The flow of those vector fields is also symmetric with respect to the straight line L. Under these hypotheses the image by the Poincaré map of the segment γ is also a spiral on the plane $z = 0$ that approaches $\mathbb{S}^2$ spiraling infinitely many times. The spiraling property of $\pi(\gamma)$ for the vector fields considered in (5), which is needed in order to guarantee the existence of infinitely many symmetric periodic orbits, is due to the fact that e^+ and e^- are foci, whereas in this paper it is only due to the presence of the semistable periodic orbit Λ.

Notice that if the vector field X satisfies conditions (C_1)–(C_5), but we consider that in condition (C_1) the singular points e^+ and e^- are foci on $\mathbb{S}^2$ instead of nodes, then by using similar arguments than for the case where the points e^+ and e^- are nodes we can prove that it also possesses infinitely many symmetric periodic orbits near $\mathcal{L}$. This case is not considered in this paper because, as we have mentioned, in such a case the existence of the semistable periodic orbit Λ would not be really necessary in order to guarantee the existence of infinitely many symmetric periodic orbits.

We note that when we perturb our model by destroying the periodic orbit Λ but not the symmetry of the flow of the system with respect to the y–axis, if the perturbation is sufficiently small some of the periodic orbits persist (the ones which are more far from the loop), whereas perhaps infinitely many of them disappear (the ones which are more close to the loop). This is due to the fact that the existence of such periodic orbits become from the intersection of the image of the y–axis through the Poincaré map with itself, and as we shall see in the proof such image is an spiral with infinitely many turns (accumulating to Λ), which intersect the y–axis, and for a sufficiently small perturbation destroying Λ the spiral persists perhaps loosing infinitely many turns near Λ.

In Section 3 we analyze the conditions that must satisfy the coefficients of an arbitrary polynomial vector field in $\mathbb{R}^3$ of degree four after imposing conditions (C_1)–(C_5), i.e. the hypotheses of Theorem 1. In particular we prove the following result.

Theorem 2 *Let $X = (P, Q, R)$ be the polynomial vector field*

$$
\begin{aligned}
P = & -b_1 y - a_2 xz - a_1 y^2 - b_5 x^2 y - b_5 y^3 - b_2 y z^2 + a_2 x^3 z + \\
& a_1 x^2 y^2 + (a_2 + c_2 - c_5 + c_7) x z^3 + a_2 x y^2 z + a_1 y^4 - b_3 y^2 z^2 \,, \\
Q = & \, b_1 x + a_1 xy - b_4 yz + b_5 x^3 + b_5 x y^2 + b_2 x z^2 - a_1 x^3 y + b_4 x^2 y z - \\
& a_1 x y^3 + b_3 x y z^2 + (b_4 + c_2 - c_3 + c_6 - c_7) y z^3 + b_4 y^3 z \,, \\
R = & -c_1 - c_2 + (c_1 + c_2 - c_7) x^2 + (c_1 + c_2 - c_6 + c_7) y^2 + c_1 z^2 - \\
& c_4 xz + c_7 x^4 + c_4 x^3 z + c_6 x^2 y^2 + c_5 x^2 z^2 + c_4 x z^3 + c_4 x y^2 z + \\
& (c_6 - c_7) y^4 + c_2 z^4 + c_3 y^2 z^2 \,.
\end{aligned}
\tag{1}
$$

Assume that the coefficients of X satisfy the following conditions

(i) *either $c_2 > 0$ and $c_1 + c_2 > 0$, or $c_2 < 0$ and $c_1 + 2c_2 > 0$, or $c_2 = 0$ and $c_1 > 0$,*
(ii) $b_1 + b_5 \neq 0$,
(iii) $2c_2 - c_3 - c_5 + c_6 < -|c_3 - c_5 - c_6 + 2c_7|$,
(iv) $-2c_1 - 2c_2 - c_6 > |c_6 - 2c_7|$,
(v) $|-c_3 + c_5 + c_6 - 2c_7| > 2|b_1 + b_2|$
(vi) $-a_2 - b_4 - 2c_1 - 2c_2 - c_6 > |-a_2 + b_4 + c_6 - 2c_7|$

Then the following statements hold.

(a) *The set of coefficients satisfying conditions (i)–(vi) is not empty.*
(b) *The vector field X satisfies all the assumptions of Theorem 1.*

2 Proof of Theorem 1

We shall prove with all details Theorem 1 for $n = 1$ and $n = 2$, see Proposition 3 and 4 respectively. The proof for $n \geqslant 2$, as we shall see from the proofs for $n = 1, 2$, will be almost the same.

Proposition 3 *Assume that the vector field X is defined on the closed ball $\mathbb{D}^3$ and it satisfies conditions (C_1)–(C_5), then X has infinitely many periodic orbits near the heteroclinic loop $\mathcal{L}$ that cross exactly 2 times the plane $z = 0$ during a period.*

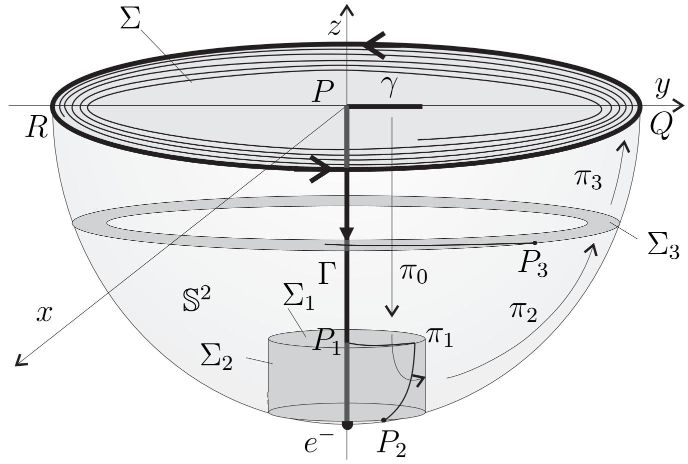

Fig. 1. The map π.

PROOF. Using the invariance of the vector field X with respect to the symmetry $(x, y, z, t) \longrightarrow (-x, y, -z, -t)$ (condition (C_5)) we have that if $\phi(t) = (x(t), y(t), z(t))$ is a solution of X, then $\psi(t) = (-x(-t), y(-t), -z(-t))$ is also a solution. This symmetry can be used in the standard way in order to obtain symmetric periodic solutions. If $\phi(t)$ is an orbit such that at $t = 0$ satisfies that $x(0) = z(0) = 0$, then the symmetric solution $\psi(t)$ satisfies that $\psi(0) = (0, y(0), 0) = \varphi(0)$. By the uniqueness theorem on the solutions of the differential system associated to X the orbits ϕ and ψ are the same. If in addition the orbit $\phi(t)$ satisfies that $\phi(\tau) = (0, y(\tau), 0)$ for some $\tau > 0$, and there is no $0 < t < \tau$ such that $x(t)$ and $z(t)$ are simultaneously zero, then the orbit $\phi = \psi$ must be a periodic orbit of period 2τ. In other words if an orbit intersects the line of symmetry L in two different points, then it is a periodic orbit.

Let $\Sigma = \{(x, y, z) \in \mathbb{D}^3 : z = 0\}$, let $\Sigma_0 = \{(x, y, z) \in \mathbb{D}^3 : z = 0, x^2 + y^2 \leqslant \varepsilon_1\}$ for a fixed $\varepsilon_1 > 0$ sufficiently small and let $\gamma = \{(0, y, 0) \in L : y > 0\} \cap \Sigma_0$. We consider a small topological cylinder in a neighbourhood of the singular point $e^- = \{(0, 0, -1)\}$ with base on $\mathbb{S}^2$ and boundaries Σ_1 and Σ_2 with $\Sigma_1 = \{(x, y, z) \in \mathbb{D}^3 : 0 < x^2 + y^2 \leqslant \varepsilon_2^2, z = -1 + \varepsilon_3\}$, and $\Sigma_2 = \{(x, y, z) \in \mathbb{D}^3 : x^2 + y^2 = \varepsilon_2^2, -1 < z \leqslant -1 + \varepsilon_3\}$, with $\varepsilon_i > 0$ sufficiently small for $i = 2, 3$. Finally we consider the section $\Sigma_3 = \{(x, y, z) \in \mathbb{D}^3 : 1 - \varepsilon_5 \leqslant x^2 + y^2 + \varepsilon_4^2 < 1, z = -\varepsilon_4\}$, with $\varepsilon_i > 0$ sufficiently small for $i = 4, 5$, see Fig. 1.

We define a map $\pi : \Sigma_0 \longrightarrow \Sigma$ by $\pi = \pi_3 \circ \pi_2 \circ \pi_1 \circ \pi_0$, where $\pi_0 : \Sigma_0 \to \Sigma_1$, $\pi_1 : \Sigma_1 \to \Sigma_2$, $\pi_2 : \Sigma_2 \to \Sigma_3$, and $\pi_3 : \Sigma_3 \to \Sigma$, in the following way. We denote by $\varphi(t, q)$ the flow generated by system X, satisfying $\varphi(0, q) = q$. For all i the map π_i from the domain A to the image B; i.e. $\pi_i : A \to B$, is defined by $\pi_i(q) = p$, where $q \in A$ and p is the point at which the orbit $\varphi(t, q)$ intersects B for the first positive time. Now we prove that the maps π_i are well defined when ε_i are sufficiently small, and consequently π is well defined for ε_1 sufficiently small. We also analyze the image by π of the segment γ.

Since $W^u(e^+) = W^s(e^-) = \Gamma$ (see condition (C_3)), the flow of X on the orbit Γ goes in the decreasing direction of the z–axis. By the continuity of the flow φ with respect to initial conditions, if $q \in \Sigma_0$ is sufficiently close to the point $P = \Sigma \cap \Gamma$, then the orbit $\varphi(t,q)$ is close to the orbit Γ for all t in a finite interval of time. Since the orbit Γ expends a finite time for going from the point P to the point $P_1 = (0,0,-1+\varepsilon_3)$, we can guarantee that if ε_1 is sufficiently small, then $\varphi(t,q)$ intersects Σ_1 for all $q \in \Sigma_0$. Consequently the map π_0 is well defined and the image by π_0 of γ is an arc on Σ_1 having P_1 as an endpoint (see Fig. 1).

From conditions (C_2) and (C_3) we have that $P_1 \in \Gamma = W^s(e^-)$ and $W^u(e^-) = \mathbb{S}^- \setminus \{e^-\}$. So if $q \in \Sigma_1$ is sufficiently close to P_1, that is, ε_2 is sufficiently small, then $\varphi(t,q)$ intersects Σ_2 and consequently π_1 is well defined. Moreover the image by π_1 of $\pi_0(\gamma) \cap \Sigma_1$ is a curve on Σ_2 that approaches a point $P_2 \in \mathbb{S}^2 \cap \{(x,y,z) \in \mathbb{D}^3 \ : \ x^2+y^2 = \varepsilon_2, z < 0\}$ when on γ we approach to P (see again Fig. 1).

Since $W^u(e^-) = W^s(\Lambda) = \mathbb{S}^- \setminus \{e^-\}$, if $q \in \Sigma_2$ is sufficiently close to P_2, that is, ε_3 is sufficiently small, then the orbit $\varphi(t,q)$ is close to the orbit $\varphi(t,P_2)$ for all t in a finite interval of time. The orbit $\varphi(t,P_2)$ intersects Σ_3 at a point $P_3 \in \mathbb{S}^2 \cap \{(x,y,z) \in \mathbb{D}^3 \ : \ z = -\varepsilon_4\}$ after a finite positive time because $\varphi(t,P_2) \subset W^s(\Lambda)$. Therefore π_2 is well defined. Moreover the image by π_2 of $\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2$ is a curve on Σ_3 that approaches to P_3 when on γ we approach to P (see again Fig. 1).

Let U be the small neighbourhood defined in condition (C_4). Clearly if ε_4 and ε_5 are sufficiently small, then $\Sigma_3 \subset U$. From condition (C_4), the flow of X on U satisfies that $\dot z > 0$. This together with the fact that $W^s(\Lambda) \cap \mathrm{Int}(\mathbb{D}^3) = \emptyset$ and $W^u(\Lambda) \cap \mathrm{Int}(\mathbb{D}^3) = \emptyset$ (because $W^s(\Lambda) = \mathbb{S}^- \setminus \{e^-\}$ and $W^u(\Lambda) = \mathbb{S}^+ \setminus \{e^+\}$) implies that if ε_4 and ε_5 are sufficiently small, then the orbit of a point $q \in \Sigma_3$ intersects Σ. Consequently π_3 is well defined. Now we analyze the image by π_3 of $\pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3$. The orbit $\varphi(t,P_3)$ tends to the periodic orbit Λ as $t \to \infty$. In fact, $\varphi(t,P_3)$ is a spiral on $\mathbb{S}^-$ which rotates infinitely many times tending to Λ. Moreover the different turns of the spiral accumulate to Λ. If $q \in \pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3$ is sufficiently close to the point P_3, then by the continuity of the flow φ with respect to initial conditions, the orbit $\varphi(t,q)$ is close to the orbit $\varphi(t,P_3)$ for all t in a finite interval of time that we denote by I_q. Moreover the interval of time I_q tends to infinity as q tends to P_3. So the number of turns near $\mathbb{S}^2$ and around the z–axis of the orbit $\varphi(t,q)$ before crossing Σ tends to infinity as q tends to P_3. Assume that $q_1, q_2 \in \pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3$ are such that $\varphi(t,q_1)$ (respectively, $\varphi(t,q_2)$) cross Σ after doing exactly n (respectively, $n+1$) turns around the z–axis. The orbits associated to the points on $\pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3$ between q_1 and q_2 will cross Σ after doing n turns but before completing the turn $n+1$. So by continuity, the image by π_3 of the points on $\pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3$

between q_1 and q_2 gives a complete turn around the z–axis. Therefore, the image by π_3 of $\pi_2(\pi_1(\pi_0(\gamma)\cap\Sigma_1)\cap\Sigma_2)\cap\Sigma_3$ is a spiral on Σ that approaches to Λ by giving infinitely many turns around the z–axis when on γ we approach to P (see again Fig. 1).

We have just seen that we can find $\varepsilon_i > 0$ for $i = 1, \ldots, 5$ such that the maps π_0, π_1, π_2 and π_3 are well defined. Moreover we can take $\varepsilon_3 = \overline{\varepsilon}_3 > 0$ small enough so that $\pi_2(\Sigma_2) \subset \Sigma_3$. Fixed $\varepsilon_3 = \overline{\varepsilon}_3$ we can take $\varepsilon_2 = \overline{\varepsilon}_2 > 0$ small enough so that $\pi_1(\Sigma_1) \subset \Sigma_2$. Finally fixed $\varepsilon_2 = \overline{\varepsilon}_2$ and $\varepsilon_3 = \overline{\varepsilon}_3$, we can take $\varepsilon_1 = \overline{\varepsilon}_1 > 0$ small enough so that $\pi_0(\Sigma_0) \subset \Sigma_1$. In short if $\varepsilon_1 = \overline{\varepsilon}_1$, then the map $\pi = \pi_3 \circ \pi_2 \circ \pi_1 \circ \pi_0$ is well defined and the image $\pi(\gamma)$ is a spiral on Σ that approaches to $\mathbb{S}^2$, when we approach to P, by giving infinitely many turns around the z–axis.

We note that $\pi(\gamma)$ intersects the line of symmetry L infinitely many times near the point Q, and infinitely many times near the point R. Since the points of γ belong to the line of symmetry, those intersection points correspond to orbits of X that cross the line of symmetry at two different points. That is, they correspond to symmetric periodic orbits. By the construction, these periodic orbits cross exactly 2 times the plane $z = 0$. $\square$

Notice that if we consider the image by π of the segment $\gamma' = \{(0, y, 0) \in L : y < 0\} \cap \Sigma_0$ instead of the segment γ, then we can repeat the arguments used in the proof of Proposition 3 and we also obtain a family of infinitely many symmetric periodic orbits near $\mathcal{L}$ that cross exactly 2 times the plane $z = 0$ during a period. This family is different than the one obtained in Proposition 3.

Proposition 4 *Assume that the vector field X is defined on the closed ball $\mathbb{D}^3$ and it satisfies conditions (C_1)–(C_5), then X has infinitely many periodic orbits near the heteroclinic loop $\mathcal{L}$ that cross exactly 4 times the plane $z = 0$ during a period.*

PROOF. We consider γ, Σ, Σ_0, Σ_1, Σ_2 and Σ_3 defined as in the proof of Proposition 3. We define $\Sigma_4 = s(\Sigma_3)$, $\Sigma_5 = s(\Sigma_2)$, and $\Sigma_6 = s(\Sigma_1)$, where s is the symmetry defined in condition (C_5) (see Fig. 2).

We define a map $\Pi : \Sigma_0 \longrightarrow \Sigma$ by $\Pi = \pi_7 \circ \pi_6 \circ \pi_5 \circ \pi_4 \circ \pi_3 \circ \pi_2 \circ \pi_1 \circ \pi_0$, where π_0, π_1, π_2 and π_3 are defined as in the proof of Proposition 3, and $\pi_4 : \Sigma \to \Sigma_4$, $\pi_5 : \Sigma_4 \to \Sigma_5$, $\pi_6 : \Sigma_5 \to \Sigma_6$, $\pi_7 : \Sigma_6 \to \Sigma$. As in Proposition 3, for all i the map π_i from the domain A to the image B; i.e. $\pi_i : A \to B$, is defined by $\pi_i(q) = p$, where $q \in A$ and p is the point at which the orbit $\varphi(t, q)$ intersects B for the first positive time. Then, by using the invariance of the flow of X under the symmetry s, we have that $\pi_4 = s \circ \pi_3^{-1} \circ s$, $\pi_5 = s \circ \pi_2^{-1} \circ s$,

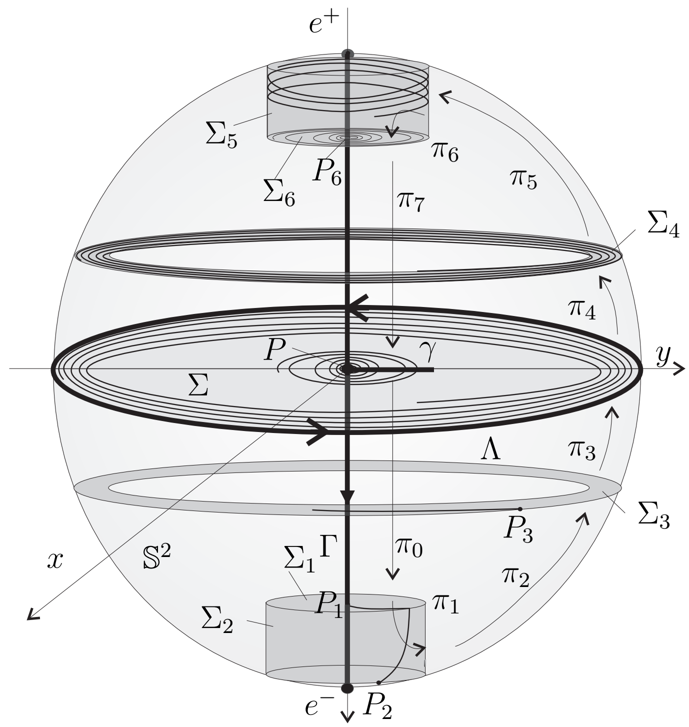

Fig. 2. The map Π.

$\pi_6 = s \circ \pi_1^{-1} \circ s$, and $\pi_7 = s \circ \pi_0^{-1} \circ s$. Clearly, if ε_1 is sufficiently small, then Π is well defined. Now we analyze the image of γ under Π.

Assume that $q_1, q_2 \in \pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3$ are such that $\varphi(t, q_1)$ (respectively, $\varphi(t, q_2)$) cross Σ after doing exactly n (respectively, $n+1$) turns around the z–axis, and that $\varphi(t, q_1) \cap \Sigma = p_1 \in L$ (respectively, $\varphi(t, q_2) \cap \Sigma = p_2 \in L$). The existence of such points follows from the proof of Proposition 3. Notice that $\varphi(t, q_1)$ and $\varphi(t, q_2)$ are symmetric orbits because $p_1 \in L$ and $p_2 \in L$. Since $\Sigma_4 = s(\Sigma_3)$ and $\varphi(t, q_1)$ (respectively, $\varphi(t, q_2)$) is a symmetric orbit that gives n (respectively, $n + 1$) turns around the z–axis when it goes from Σ_3 to Σ, $\varphi(t, p_1)$ (respectively, $\varphi(t, p_2)$) intersects Σ_4 after doing n (respectively, $n+1$) additional turns around the z–axis. Thus $\varphi(t, q_1)$ (respectively, $\varphi(t, q_2)$) intersects Σ_4 after doing $2n$ (respectively, $2n + 2$) turns around the z–axis. As in Proposition 3, this implies that the image by $\pi_4 \circ \pi_3$ of the points on $\pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3$ between q_1 and q_2 gives two complete turns around the z–axis. Therefore the image by π_4 of $\pi_3(\pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3)$ is a spiral on Σ_4 that approaches to $\mathbb{S}^2$ by giving infinitely many turns around the z–axis when on γ we approach to P (see again Fig. 2).

Since the orbits on $\mathbb{S}^2 \cap \{(x, y, z) \in \mathbb{D}^3 : z = \varepsilon_4\}$ expend a finite time for going form Σ_4 to $\mathbb{S}^2 \cap \{(x, y, z) \in \mathbb{D}^3 : x^2 + y^2 = \varepsilon_2, z > 0\}$, we can guarantee that the image by π_5 of $\pi_4(\pi_3(\pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3)) \cap \Sigma_4$ is a spiral on Σ_5 that approaches to $\mathbb{S}^2$ by giving infinitely many turns around the z–axis when on γ we approach to P.

Now we prove that the fact that e^+ is an hyperbolic singular point which is a node on $\mathbb{S}^2$ with $W^s(e^+) = \mathbb{S}^+ \setminus \{e^+\}$ and such that $W^u(e^+) = \Gamma$, implies that the image by π_6 of $\pi_5(\pi_4(\pi_3(\pi_2(\pi_1(\pi_0(\gamma)\cap\Sigma_1)\cap\Sigma_2)\cap\Sigma_3))\cap\Sigma_4)\cap\Sigma_5$ is a spiral on Σ_6 that approaches the point $P_6 = (0, 0, 1 - \varepsilon_3)$ by giving infinitely many turns around the z–axis when on γ we approach to P.

Since e^+ is a hyperbolic singular point which is a node on $\mathbb{S}^2$ with $W^s(e^+) = \mathbb{S}^+ \setminus \{e^+\}$ and such that $W^u(e^+) = \Gamma$, by the Hartman Theorem, the flow of X in a sufficiently small neighborhood of e^+ is topologically conjugate to the flow of the linear differential system given by either

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2}$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{3}$$

for some $\lambda_1, \lambda_2 < 0$ and $\lambda_3 > 0$. Therefore if ε_2 and ε_3 are sufficiently small, then the image by π_6 of a curve $\zeta = \{(x, y, z) \in \Sigma_5 \ : \ x = \varepsilon_2 \cos\theta, y = \varepsilon_2 \sin\theta, z = z_0(\theta), \theta \in [0, 2\pi]\}$ is topologically conjugate to the image by $\overline{\pi} : \overline{\Sigma}_5 \to \overline{\Sigma}_6$ of the curve $\overline{\zeta} = \{(x, y, z) \in \overline{\Sigma}_5 \ : \ x = \varepsilon_2 \cos\theta, y = \varepsilon_2 \sin\theta, z = z_0(\theta) - 1, \theta \in [0, 2\pi]\}$. Here $\overline{\Sigma}_5 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = \varepsilon_2^2, z \in [-\varepsilon_3, 0)\}$, $\overline{\Sigma}_6 = \{(x, y, z) \in \mathbb{R}^3 \ : \ 0 < x^2 + y^2 \leqslant \varepsilon_2^2, z = -\varepsilon_3\}$ and $\overline{\pi} : \overline{\Sigma}_5 \to \overline{\Sigma}_6$ is defined by $\overline{\pi}(q) = p$, where $q \in \overline{\Sigma}_5$ and p is the point at which the orbit of the flow generated by either system (2) or system (3), $\overline{\varphi}(t, q)$, intersects $\overline{\Sigma}_6$ for the first positive time.

Next we compute the image by $\overline{\pi}$ of the circle $\overline{\zeta}_{\overline{z}_0} = \{(x, y, z) \in \overline{\Sigma}_5 \ : \ x = \varepsilon_2 \cos\theta, y = \varepsilon_2 \sin\theta, z = \overline{z}_0, \theta \in [0, 2\pi]\}$ for some fixed $\overline{z}_0 \in [-\varepsilon_3, 0)$. After some computations we get that the solution of (2) with initial conditions $x(0) = \varepsilon_2 \cos\theta$, $y(0) = \varepsilon_2 \sin\theta$ and $z(0) = \overline{z}_0$ is given by

$$x(t) = \varepsilon_2 \cos\theta e^{\lambda_1 t}, \qquad y(t) = \varepsilon_2 \sin\theta e^{\lambda_2 t}, \qquad z(t) = \overline{z}_0 e^{\lambda_3 t}.$$

This solution intersects $\overline{\Sigma}_6$ at $t = \ln\left(-\varepsilon_3/\overline{z}_0\right)/\lambda_3$. So the image $\overline{\pi}(\overline{\zeta}_{\overline{z}_0})$ is a curve on $\overline{\Sigma}_6$ such that

$$x = \varepsilon_2 \cos\theta \left(\frac{-\varepsilon_3}{\overline{z}_0}\right)^{\frac{\lambda_1}{\lambda_3}}, \qquad y = \varepsilon_2 \sin\theta \left(\frac{-\varepsilon_3}{\overline{z}_0}\right)^{\frac{\lambda_2}{\lambda_3}}. \tag{4}$$

Notice that expression (4) gives the parametric equations of an ellipse centered at the origin with semimajor axis $a(\overline{z}_0) = \varepsilon_2\left(-\varepsilon_3/\overline{z}_0\right)^{\lambda_1/\lambda_3}$ and semiminor axis

$b(\overline{z}_0) = \varepsilon_2 \left(-\varepsilon_3/\overline{z}_0\right)^{\lambda_2/\lambda_3}$. On the other hand, if $0 > \overline{z}_0 > \overline{z}_1$, then $a(\overline{z}_0) < a(\overline{z}_1)$ and $b(\overline{z}_0) < b(\overline{z}_1)$ because $\lambda_1, \lambda_2 < 0$ and $\lambda_3 > 0$. This implies that when the linearization of X near the singular point e^+ diagonalizes, the image by $\overline{\pi}$ of a spiral on $\overline{\Sigma}_5$ that approaches to $z = 0$ by giving infinitely many turns around the z–axis is a spiral on $\overline{\Sigma}_6$ that approaches the point $(0, 0, -\varepsilon_3)$ by giving infinitely many turns around the z–axis.

The solution of (3) with initial conditions $x(0) = \varepsilon_2 \cos\theta$, $y(0) = \varepsilon_2 \sin\theta$ and $z(0) = \overline{z}_0$ is given by

$$x(t) = \varepsilon_2 \cos\theta e^{\lambda_1 t}, \qquad y(t) = \varepsilon_2 e^{\lambda_1 t}(t\cos\theta + \sin\theta), \qquad z(t) = \overline{z}_0 e^{\lambda_3 t}.$$

As above, this solution intersects $\overline{\Sigma}_6$ at $t = \ln\left(-\varepsilon_3/\overline{z}_0\right)/\lambda_3$. So the image $\overline{\pi}(\overline{\zeta}_{\overline{z}_0})$ is a curve on $\overline{\Sigma}_6$ such that

$$x = \varepsilon_2 \cos\theta \left(\frac{-\varepsilon_3}{\overline{z}_0}\right)^{\frac{\lambda_1}{\lambda_3}}, \quad y = \varepsilon_2 \left(\frac{-\varepsilon_3}{\overline{z}_0}\right)^{\frac{\lambda_1}{\lambda_3}} \left(\frac{\ln\left(-\varepsilon_3/\overline{z}_0\right)}{\lambda_3}\cos\theta + \sin\theta\right). \tag{5}$$

From (5) we have that

$$\cos\theta = \frac{x}{\varepsilon_2}\left(-\frac{\varepsilon_3}{\overline{z}_0}\right)^{-\frac{\lambda_1}{\lambda_3}}, \qquad \sin\theta = \frac{1}{\varepsilon_2}\left(-\frac{\varepsilon_3}{\overline{z}_0}\right)^{-\frac{\lambda_1}{\lambda_3}}\left[y - x\frac{\ln\left(-\varepsilon_3/\overline{z}_0\right)}{\lambda_3}\right].$$

Since $\cos^2\theta + \sin^2\theta = 1$, the curve defined by (5) can be written as

$$\frac{x^2\left(\lambda_3^2 + \ln^2\left(-\frac{\varepsilon_3}{\overline{z}_0}\right)\right)\left(-\frac{\varepsilon_3}{\overline{z}_0}\right)^{-\frac{2\lambda_1}{\lambda_3}}}{\varepsilon_2^2\lambda_3^2} + \frac{y^2\left(-\frac{\varepsilon_3}{\overline{z}_0}\right)^{-\frac{2\lambda_1}{\lambda_3}}}{\varepsilon_2^2} - \frac{2xy\ln\left(-\frac{\varepsilon_3}{\overline{z}_0}\right)\left(-\frac{\varepsilon_3}{\overline{z}_0}\right)^{-\frac{2\lambda_1}{\lambda_3}}}{\varepsilon_2^2\lambda_3} = 1.$$

This is the equation of an ellipse centered at the origin rotated by an angle $\phi = \cot^{-1}\left(\ln(-\varepsilon_3/\overline{z}_0)/(2\lambda_3)\right)/2$ and having semiaxis

$$a(\overline{z}_0) = \sqrt{\frac{2\varepsilon_2^2\left(-\varepsilon_3/\overline{z}_0\right)^{\frac{2\lambda_1}{\lambda_3}}\lambda_3^2}{2\lambda_3^2 + \left(1 + \sqrt{1 + 4\lambda_3^2/\ln^2\left(-\varepsilon_3/\overline{z}_0\right)}\right)\ln^2\left(-\varepsilon_3/\overline{z}_0\right)}},$$

$$b(\overline{z}_0) = \sqrt{\frac{2\varepsilon_2^2\left(-\varepsilon_3/\overline{z}_0\right)^{\frac{2\lambda_1}{\lambda_3}}\lambda_3^2}{2\lambda_3^2 + \left(1 - \sqrt{1 + 4\lambda_3^2/\ln^2\left(-\varepsilon_3/\overline{z}_0\right)}\right)\ln^2\left(-\varepsilon_3/\overline{z}_0\right)}}.$$

Since $\lambda_1, \lambda_2 < 0$ and $\lambda_3 > 0$, we can prove after some computations that $a(\overline{z}_0)$ is decreasing for all $\overline{z}_0 < 0$ and $b(\overline{z}_0)$ is decreasing for all $\overline{z}_0 \in (-e^{\sqrt{\lambda_3^2 - 4\lambda_1^2\lambda_3^2}/\lambda_1}\varepsilon_3, 0)$. This implies that when the linearization of X near the point e^+ do not diagonalize, the image by $\overline{\pi}$ of a spiral on $\overline{\Sigma}_5$ that approaches to $z = 0$ by giving infinitely many turns around the z–axis is still a spiral on $\overline{\Sigma}_6$ that approaches the point $(0, 0, -\varepsilon_3)$ by giving infinitely many turns around the z–axis.

Finally, since the orbit Γ expends a finite time for going from Σ_6 to Σ, the image by π_7 of $\pi_6(\pi_5(\pi_4(\pi_3(\pi_2(\pi_1(\pi_0(\gamma) \cap \Sigma_1) \cap \Sigma_2) \cap \Sigma_3)) \cap \Sigma_4) \cap \Sigma_5) \cap \Sigma_6$ is a spiral on Σ that approaches the point P by giving infinitely many turns around the z–axis when on γ we approach to P (see again Fig. 2).

In short, $\Pi(\gamma)$ is a spiral on Σ that approaches the point P by giving infinitely many turns around the z–axis when on γ we approach to P. The points of $\Pi(\gamma) \cap L$ give the infinitely many symmetric periodic orbits. It is easy to see by using the symmetry s that the points of $\Pi(\gamma) \cap \gamma$ correspond to the symmetric periodic orbits found in Proposition 3, which cross exactly 2 times the plane $z = 0$ during a period, whereas the points of $\Pi(\gamma) \cap \gamma'$ correspond to symmetric periodic orbits that cross exactly 4 times the plane $z = 0$ during a period. $\square$

We could also analyze the image by Π of the segment γ', but $\Pi(\gamma') \cap L$ does not give new symmetric periodic orbits of X that cross exactly 4 times the plane $z = 0$ during a period because the points of $\Pi(\gamma') \cap \gamma'$ correspond to symmetric periodic orbits that cross exactly 2 times the plane $z = 0$ during a period and the points of $\Pi(\gamma') \cap \gamma$ coincide with the points of $\Pi(\gamma) \cap \gamma'$.

We remark that if we consider that the equilibrium points e^+ and e^- are foci instead of nodes, then Propositions 3 and 4 also hold. In particular, the proof of Proposition 3 when e^+ and e^- are foci is the same but taking into account that the image by π_1 of $\pi_0(\gamma) \cap \Sigma_1$ is a spiral on Σ_2 that approaches $\mathbb{S}^2$ by giving infinitely many turns around the z–axis when on γ we approach to P. The proof of Proposition 4 when e^+ and e^- are foci is also the same but now the preservation of the spiralling structure near the point e^+ is an immediate consequence of the fact that e^+ is a focus.

2.1 Extension of the proof for arbitrary n

Using similar arguments than the ones used in Propositions 3 and 4, we see that the image by π of $\Pi(\gamma)$ is a spiral on Σ that approaches to Λ spiraling infinitely many times when on γ we approach to P. Clearly, $\pi(\gamma) \cap L \subset \pi(\Pi(\gamma)) \cap L$, so $\pi(\Pi(\gamma)) \cap L$ contains the intersection points corresponding to the symmetric periodic orbits found in Proposition 3, which cross exactly 2 times the plane $z = 0$ during a period. Next we prove that $\pi(\Pi(\gamma)) \cap L$ contains infinitely many additional intersection points that correspond to symmetric periodic orbits that cross exactly 6 times the plane $z = 0$ during a period.

Let $q_1, q_2 \in \gamma$ be such that $\pi(q_1) \in L \cap \{y > 0\}$ and $\pi(q_2) \in L \cap \{y > 0\}$ and that the orbit $\varphi(t, q_1)$ (respectively, $\varphi(t, q_2)$) intersects Σ for the first time after doing exactly n (respectively, $n+1$) turns around the z–axis. As we have seen, the image by π of the segment $[q_2, q_1]$ is an arc on Σ close to Λ that gives a complete turn around the z–axis. Notice that the image by π of the interval $[q_2, q_1]$ gives three symmetric periodic orbits that cross exactly 2 times the plane $z = 0$ during a period, the two associated to the points $\pi(q_1)$ and $\pi(q_2)$ and another one associated to the point $\pi([q_2, q_1]) \cap L \cap \{y < 0\}$. By means of the symmetry s, the orbit $\varphi(t, q_1)$ (respectively, $\varphi(t, q_2)$) intersects Σ for the second time after doing exactly $2n$ (respectively, $2n + 2$) turns around the z–axis. So the image by Π of the segment $[q_2, q_1]$ is an arc on Σ close to P that gives two complete turns around the z–axis. The image by Π of $[q_2, q_1]$ intersects L at 5 different points, the three points $\Pi([q_2, q_1]) \cap \gamma$ correspond to the three symmetric periodic orbits that cross exactly 2 times the plane $z = 0$ during a period given by $\pi([q_2, q_1]) \cap L$, whereas the two points $\Pi([q_2, q_1]) \cap \gamma'$ correspond to symmetric periodic orbits that cross exactly 4 times the plane $z = 0$.

In a similar way, wee see that the orbit $\varphi(t, q_1)$ (respectively, $\varphi(t, q_2)$) intersects Σ for the third time after doing exactly $3n$ (respectively, $3n + 3$) turns around the z–axis. So the image by $\pi \circ \Pi$ of the segment $[q_2, q_1]$ is an arc on Σ close to Λ that gives three complete turns around the z–axis. Notice that the image by $\pi \circ \Pi$ of $[q_2, q_1]$ intersects L at 7 different points, 3 of them correspond to the three symmetric periodic orbits that cross exactly 2 times the plane $z = 0$ during a period given by $\pi([q_2, q_1]) \cap L$, whereas the other 4 correspond to symmetric periodic orbits that cross exactly 6 times the plane $z = 0$. In short, $\pi(\Pi(\gamma)) \cap L$ gives infinitely many points that correspond to symmetric periodic orbits that cross exactly 6 times the plane $z = 0$ during a period.

Proceeding in a similar way, we have that the orbit $\varphi(t, q_1)$ (respectively, $\varphi(t, q_2)$) intersects Σ for the fourth time after doing exactly $4n$ (respectively, $4n + 4$) turns around the z–axis. So the image by Π^2 of the segment $[q_2, q_1]$ is an arc on Σ close to P that gives four complete turns around the z–axis, consequently it intersects 9 times the line of symmetry L. It is easy to see that 3 of these intersection points correspond to symmetric periodic orbits that cross exactly 2 times the plane $z = 0$ during a period, 2 of them correspond to symmetric periodic orbits that cross exactly 4 times the plane $z = 0$ during a period, and the remaining ones correspond to symmetric periodic orbits that cross exactly 8 times the plane $z = 0$ during the period. Therefore $\Pi^2(\gamma) \cap L$ gives infinitely many points that correspond to symmetric periodic orbits that cross exactly 8 times the plane $z = 0$ during a period.

The prove of Theorem 1 for all $n > 4$ can be done in a similar way by taking into account that the image by Π^n of the segment $[q_2, q_1]$ is an arc on Σ close to P that gives $2n$ complete turns around the z–axis, and the image by $\pi \circ \Pi^n$ is an arc on Σ close to Λ that gives $2n + 1$ complete turns around the z–axis.

In Figure 3 we illustrate the intertwined structure of the points on the curve γ that provide symmetric periodic orbits that intersect exactly $2n$ times the

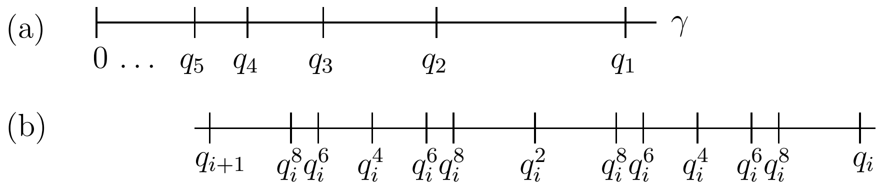

Fig. 3. For a fixed value of n sufficiently large, for all $i \in \mathbb{N}$, the points $q_i, q_{i+1} \in \gamma$ represent initial conditions of symmetric periodic orbits that cross 2 times the plane $z = 0$ during a period and such that the orbit $\varphi(t, q_i)$ (respectively, $\varphi(t, q_{i+1})$) intersects Σ for the first time after doing exactly $n+i-1$ (respectively, $n+i$) turns around the z–axis. In (a) we illustrate the distribution on γ of those points. In (b), for a fixed i, we illustrate the distribution of the initial conditions on the interval $[q_{i+1}, q_i]$ of the symmetric periodic orbits that cross exactly 2, 4, 6 and 8 times de plane $z = 0$ during a period. The notation q_i^k represents the initial condition of a symmetric periodic orbit that cross exactly k times the plane $z = 0$.

plane $z = 0$ for $n = 1, \ldots, 4$.

3 Proof of Theorem 2

Let X be the polynomial vector field in $\mathbb{R}^3$ given by (1). First we prove that if the coefficients of X satisfy conditions (i)–(vi), then X satisfies conditions (C_1)–(C_5).

It is easy to check that the flow of X is invariant under the time–reversibility symmetry s, so condition (C_5) is satisfied.

Clearly e^+ and e^- are singular points of X. From condition (C_3), we have that $W^u(e^+) = W^s(e^-) = \Gamma$. This condition is satisfied if and only if Γ is invariant under the flow of X, it does not contain any singular point and the flow of X on Γ goes in the decreasing direction of the z–axis. The vector field X restricted to the z–axis is given by

$$\dot{x} = 0\,, \qquad \dot{y} = 0\,, \qquad \dot{z} = c_2 z^4 + c_1 z^2 - c_1 - c_2\,.$$

So the z–axis is invariant under the flow of X. The singular points of X on the z–axis are given by the roots of equation $c_2 z^4 + c_1 z^2 - c_1 - c_2 = 0$. If $c_2 \neq 0$, then the roots of this equation are $z = \pm 1$ and $z = \pm\sqrt{-(c_1 + c_2)/c_2}$, whereas if $c_2 = 0$, then they are $z = \pm 1$. Clearly if $c_2 = 0$ and $c_1 > 0$, then Γ does not contain any singular point and the flow of X on Γ goes in the decreasing direction of the z–axis. Now we analyze the case $c_2 \neq 0$. None of the singular points $z = \pm\sqrt{-(c_1 + c_2)/c_2}$ should belong to Γ, thus either $-(c_1 + c_2)/c_2 < 0$ or $-(c_1 + c_2)/c_2 > 1$. Moreover the flow of X on Γ must go in the decreasing direction of the z–axis, thus $c_1 + c_2 > 0$. If $c_2 > 0$ and $c_1 + c_2 > 0$, then $-(c_1 + c_2)/c_2 < 0$, and if $c_2 < 0$ and $c_1 + 2c_2 > 0$, then $c_1 + c_2 > -c_2 > 0$

which implies that $-(c_1+c_2)/c_2 > 1$. In short, since the coefficients c_1 and c_2 satisfy condition (i), condition (C_3) is satisfied.

The eigenvalues of the vector field X at the singular point e^- are

$$\lambda_1 = -2(c_1+2c_2)\,, \qquad \lambda_{2,3} = \frac{1}{2}\left(-2c_2+c_3+c_5-c_6 \pm \sqrt{\Delta}\right)\,,$$

with eigenvectors

$$\vec{v}_{\lambda_1} = (0,0,1)\,, \qquad \vec{v}_{\lambda_{2,3}} = \left(\frac{-c_3+c_5+c_6-2c_7 \pm \sqrt{\Delta}}{2(b_1+b_2)}, 1, 0\right)\,,$$

where $\Delta = (-c_3+c_5+c_6-2c_7)^2 - 4(b_1+b_2)^2$.

From condition (i) we have that $\lambda_1 < 0$, and from conditions (iii) and (v) we have that $\lambda_{2,3} > 0$. Therefore e^- is an unstable node on $\mathbb{S}^2$. In a similar way we can prove that e^+ is a stable node on $\mathbb{S}^2$. So condition (C_1) is satisfied.

Now we study the flow of X on the invariant sphere $\mathbb{S}^2$ by using spherical coordinates $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$ where $r \in [0,+\infty)$, $\theta \in [-\pi/2, \pi/2]$ and $\phi \in [0, 2\pi)$. In these coordinates, the vector field X restricted to the sphere $\mathbb{S}^2$ (that is, restricted to $r=1$) is given by

$$\begin{aligned}
\dot{r} &= 0\,,\\
\dot{\theta} &= -\frac{\cos\theta\sin^2\theta}{2}\left(2c_2-c_3-c_5+c_6+(c_3-c_5-c_6+2c_7)\cos(2\phi)\right)\,,\\
\dot{\phi} &= \frac{1}{2}\Big(2b_1+b_2+b_5+(2b_3+2a_1)\cos\theta\sin\phi\sin^2\theta -\\
&\quad (b_2-b_5)\cos(2\theta)-(c_3-c_5-c_6+2c_7)\sin(2\phi)\sin^3\theta\Big)\,.
\end{aligned}$$

Since $\dot{r}=0$ the sphere $\mathbb{S}^2$ is invariant under the flow of X. The fact that e^- is an unstable node on $\mathbb{S}^2$ and e^+ is a stable node on $\mathbb{S}^2$ implies that $W^u(e^-) \subset \mathbb{S}^2$ and $W^s(e^+) \subset \mathbb{S}^2$. The circle $\Lambda = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1,\, z = 0\}$ in spherical coordinates is given by $r=1$ and $\theta = 0$. The vector field X restricted to $r=1$ and $\theta=0$ becomes

$$\dot{r} = 0\,, \qquad \dot{\theta} = 0\,, \qquad \dot{\phi} = b_1+b_5\,.$$

So Λ is invariant under the flow of X. Since b_1 and b_5 satisfy condition (ii), the circle Λ does not contain any singular point of X, so it is a periodic orbit. On the other hand, since c_2, c_3, c_4, c_5, and c_6 satisfy condition (iii), the derivative $\dot{\theta}$ restricted to $r=1$ is positive for all $\theta \in (-\pi/2, 0) \cup (0, \pi/2)$ and $\phi \in [0, 2\phi)$, and consequently $\mathbb{S}^- \setminus \{e^-\} \subset W^s(\Lambda)$ and $\mathbb{S}^+ \setminus \{e^+\} \subset W^u(\Lambda)$. This also implies that $W^u(e^-) = \mathbb{S}^- \setminus \{e^-\}$ and $W^s(e^+) = \mathbb{S}^+ \setminus \{e^+\}$.

Finally we study the sign of $\dot{r}$ and $\dot{\theta}$ in a small neighbourhood U in $\mathbb{D}^3$ of the periodic orbit Λ. The derivatives $\dot{r}$ and $\dot{\theta}$ when $r = 1 - \varepsilon$ and $\theta = \delta$, with $\varepsilon, \delta > 0$ sufficiently small are given by

$$
\begin{aligned}
\dot{r} = & -(a_2 + b_4 + 2c_1 + 2c_2 + c_6 + (a_2 - b_4 - c_6 + 2c_7)\cos(2\phi))\,\varepsilon\,\delta + \\
& O_3(\varepsilon, \delta)\,, \\
\dot{\theta} = & -(2c_1 + 2c_2 + c_6 - (c_6 - 2c_7)\cos(2\phi))\,\varepsilon + O_2(\varepsilon, \delta)\,.
\end{aligned}
$$

Since the coefficients of X satisfy conditions (iv) and (vi), $\dot{r} < 0$ in $U \cap \{z < 0\}$, $\dot{r} > 0$ in $U \cap \{z > 0\}$ and $\dot{\theta} > 0$ for all ε and δ sufficiently small. The fact that $\dot{\theta} > 0$ implies condition (C_4). The fact that $\dot{r} < 0$ in $U \cap \{z < 0\}$ and $\dot{r} > 0$ in $U \cap \{z > 0\}$ implies that $W^s(\Lambda) \cap \mathrm{Int}(\mathbb{D}^3) \neq \emptyset$ and $W^u(\Lambda) \cap \mathrm{Int}(\mathbb{D}^3) \neq \emptyset$. Therefore $W^s(\Lambda) = \mathbb{S}^- \setminus \{e^-\}$ and $W^u(\Lambda) = \mathbb{S}^+ \setminus \{e^+\}$.

This completes the proof of statement (b) of Theorem 2.

The set of coefficients satisfying conditions (i)–(vi) is not empty, for instance if we take $b_1 = 1$, $b_2 = 0$, $b_5 = 0$, $c_1 = 1$, $c_2 = 0$, $c_3 = 3$, $c_5 = 0$, $c_6 = -6$, $c_7 = -2$, $a_2 = 0$ and $b_4 = 0$ then conditions (i)–(vi) are satisfied. This proves statement (a) of Theorem 2.

Taking the remaining coefficients equal to zero; that is, $a_1 = 0$, $b_3 = 0$, and $c_4 = 0$ we obtain a polynomial vector field satisfying the dynamics described in Theorem 1, which is given by

$$
\begin{aligned}
\dot{x} &= -y - 2xz^3\,, \\
\dot{y} &= x - 7yz^3\,, \\
\dot{z} &= -1 + 3x^2 + 5y^2 + z^2 - 2x^4 - 6x^2y^2 - 4y^4 + 3y^2z^2\,.
\end{aligned}
$$

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