Frontshear and backshear instabilities of the mean longshore current

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Abstract. An analytical model based on Bowen and Holman [1989] is used to prove the existence of instabilities due to the presence of a second extremum of the background vorticity at the front side of the longshore current. The growth rate of the so-called frontshear waves depends primarily upon the frontshear but also upon the backshear and the maximum and the width of the current. Depending on the values of these parameters, either the frontshear or the backshear instabilities may dominate. Both types of waves have a cross-shore extension of the order of the width of the current, but the frontshear modes are localized closer to the coast than are the backshear modes. Moreover, under certain conditions both unstable waves have similar growth rates with close wave numbers and angular frequencies, leading to the possibility of having modulated shear waves in the alongshore direction. Numerical analysis performed on realistic current profiles confirm the behavior anticipated by the analytical model. The theory has been applied to a current profile fitted to data measured during the 1980 Nearshore Sediment Transport Studies experiment at Leadbetter Beach that has an extremum of background vorticity at the front side of the current. In this case and in agreement with field observations, the model predicts instability, whereas the theory based only on backshear instability failed to do so.

1. Introduction

In general, the wave-driven mean longshore current in the surf zone has a horizontal profile that increases seaward of the shoreline, reaches a maximum, and then decreases to a vanishing value beyond the breaking line. Thus the background vorticity ($\nabla \times \mathbf{u}$, where $\mathbf{u}$ is the horizontal shear of the current and $\nabla \times$ is the total mean depth) has at least one extremum seaward of the peak of the current. This is a necessary condition (Rayleigh condition) for the current to be unstable with respect to alongshore traveling perturbations so-called shear waves. Bowen and Holman [1989] (hereinafter referred to as BH) illustrated the mechanism of the shear instability by means of a simple velocity profile with only one local extremum of the background vorticity on the seaward side of the current. In this case, the instability was clearly related to the extremum of the background vorticity at the back (seaward side of the current), and their model showed a good agreement with the field observations of Oltman-Shay et al. [1989]. Following BH, most of the theoretical analysis on linear shear waves have considered current profiles with only one seaward extremum of the background vorticity and thus have been related to the backshear [see, e.g., Patrevu and Svendsen, 1992; Dodd and Thornton, 1990; Falqués and Iranso, 1994; Caballeria et al., 1998]. Laboratory experiments by Reniers et al. [1997] have also shown good agreement between measured and predicted wavenumbers and frequencies of shear waves on the basis of the backshear instability.

Dodd et al. [1992], comparing stability properties at a flat and at a barred beach, observed that for the latter more than one unstable mode may arise. Since the background vorticity of the current in this case had more than one extremum, they associated one of them to the backshear and the other to the frontshear. They claimed that the mode associated to the backshear over the bar was the fastest growing mode, while the second fastest one was the mode related to the frontshear. They therefore concluded that for barred beaches the backshear may not be so important. Also, some studies on nonlinear shear waves [see, e.g., Allen et al., 1996; Oszkan-Haller and Kirby, 1999] have considered profiles of the basic steady current with two inflexion points, although the nonlinear analysis was based on the linearly dominant mode without caring about its origin. Those studies and the present paper suggest that under some circumstances the low-frequency modulation of the shear waves could be due to the interference of the frontshear and the backshear modes.

In this paper, the existence and the properties of two instability modes, one related to the extremum of the background vorticity seaward of the peak of the current (and to the backshear (BS) mode) and the other related to the extremum seaward of the peak of the current (and to the frontshear (FS) mode) are investigated in detail. To this end a velocity profile with a maximum and two inflexion points at both sides of it is analyzed. The characteristics, wavelength, frequency, flow pattern, and conditions under which one or both modes are dominant are investigated. In order to deal with a simple analytical solution, following BH, an idealized current profile on a horizontal bottom is considered. Next, a similar analysis on realistic current and topography profiles, carried out by means of numerical simulation including bottom friction and turbulent momentum diffusion, confirms the validity of the idealized theoretical results. It is shown that under certain
conditions the frontshear is indeed dominant. Moreover, there are longshore current profiles that have frontshear and backshear waves of similar growth rates, making plausible the occurrence of a field modulated in the longitudinal direction. Finally, instability analysis of a velocity profile obtained from data measured at Leadbetter Beach [Thornton and Guza, 1986] that shows an inflexion point at the front side is performed. Results are compared with field data observations, showing good agreement with measured frequency-cyclic wave number spectra.

The paper is organized as follows. The theoretical framework is presented in section 2. The simple analytical model is developed in section 3. The numerical simulation for realistic conditions, including the comparison with Leadbetter Beach field data observations, is described in section 4. The conclusions are given in section 5.

2. Formulation

The shallow water equations for momentum and mass conservation are considered as governing equations:

\[ \frac{\partial v_i}{\partial t} + \sum_{j=1}^{2} v_j \frac{\partial v_i}{\partial x_j} + g \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i} = \frac{1}{\rho} \left( \frac{\partial}{\partial x_i} (\tau_n + \tau_b + \tau_d) \right) \]

(1)

where \( v_i \) is the horizontal velocity, \( \tau_n \) stands for the water density, \( t \) stands for time, and \( \tau_b \) and \( \tau_d \) are the bottom friction and dissipation terms from bottom friction, respectively, and turbulent lateral momentum diffusion \( \tau_d \). The wave term is calculated from the radiation stress tensor \( S_{ij} \) as

\[ \tau_n = -\frac{\partial S_{ij}}{\partial x_j}, \quad i = 1, 2. \]

(3)

The bottom shear stress is given by

\[ \tau_b = -c_d \rho \left( |u_0| + v \right) \left( |u_0| + v \right), \]

(4)

where \( u_0 \) is orbital velocity, \( c_d \) is the drag coefficient for bottom friction, and brackets \( (\cdot) \) mean temporal average in the incoming short wave period. The turbulent lateral momentum diffusion is evaluated through

\[ \tau_d = \rho \nu \left( \frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial x_i} \right), \quad i = 1, 2, \]

(5)

where \( \nu \) is the kinematic eddy viscosity.

The basic undisturbed state consists of a longshore current,

\[ v_1 = 0 \quad v_2 = V(x), \]

(6)

and a setup/setdown of the free surface,

\[ \eta = \eta_0(x), \]

(7)

which are a steady solution of (1) and (2). An effective depth is defined by

\[ h_e = h + \eta_0, \]

(8)

and the cross-shore coordinate \( x \) is shifted with respect to \( x \) in order to have \( x = 0 \) at the effective shoreline (hereinafter \( x \) will take the coordinates \( x, y = x_2 \)). Then the free surface elevation measured from the basic undisturbed state \( \eta_0 \) is

\[ \xi(x, y, t) = \eta(x, y, t) - \eta_0(x), \]

(9)

and the total depth is

\[ \zeta = h + \eta_0 + \xi = h_e + \xi. \]

(10)

To perform linear stability analysis, a small perturbation is superimposed to the basic steady flow and the free surface elevation:

\[ v_1 = u(x, y, t) = \hat{u}(x)e^{i(ky-\sigma t)}, \]

(11)

\[ v_2 = V(x) + v(x, y, t) = V(x) + \hat{v}(x)e^{i(ky-\sigma t)}, \]

(12)

where the real number \( k \) takes account of the alongshore periodicity of wavelength \( \lambda = 2\pi/k \), the real part \( \sigma \) of \( \sigma \) gives the angular frequency of the shear wave, and its imaginary part \( \sigma \) gives the growth rate. The perturbation is periodic in time with period \( T_s = 2\pi/\sigma \) and propagates in the longshore direction with celerity \( c = \sigma/k \).

By using the weak-current approximation [Dodd, 1994], using a small angle of incidence, and taking orbital velocity amplitude

\[ u_0 = \frac{\gamma}{2} \sqrt{2g\xi}, \]

(13)

where \( \gamma \) is the breaking index, the bottom friction becomes

\[ \tau_{bu} = -pc_d \frac{\gamma}{2} \sqrt{g\xi} u, \]

(14)

where \( p \) stands for the water density, \( \gamma \) stands for the depth-averaged horizontal velocity. The wind/swell wave forcing is given by \( \tau_{bu} \) and dissipation comes from bottom friction, \( \tau_{bu} \), and turbulent lateral momentum diffusion \( \tau_{bu} \). The wave term is calculated from the radiation stress tensor \( S_{ij} \) as

\[ \tau_{bu} = -\frac{\partial S_{ij}}{\partial x_j}, \quad i = 1, 2. \]

The bottom shear stress is given by

\[ \tau_b = -c_d \rho (|u_0| + v)(|u_0| + v), \]

(15)

where \( u_0 \) is orbital velocity, \( c_d \) is the drag coefficient for bottom friction, and brackets \( (\cdot) \) mean temporal average in the incoming short wave period. The turbulent lateral momentum diffusion is evaluated through

\[ \tau_{bu} = \rho \nu \left( \frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial x_i} \right), \quad i = 1, 2, \]

(16)

where \( \nu \) is the kinematic eddy viscosity.

The basic undisturbed state consists of a longshore current,

\[ v_1 = 0 \quad v_2 = V(x), \]

(17)

and a setup/setdown of the free surface,

\[ \eta = \eta_0(x), \]

(18)

which is an eigenfunction due to the longshore field has been extended by the perturbation \( \eta_0(x) \) and \( \sigma \).

3. Extension

The existence of a background current \( V(x) \) the way it can be investigated in the description of a longshore current incident on a uniformly long extent sea (Figure 1). In a region \( 0 \leq \delta \leq L < \infty \) the current increases from the maximum value \( V(x) \) to \( \eta_0 \) and extends seawards.

The extension of a current function that is of value \( V(x)/h \) value \( V(x)/h \) is only one extensional eigenfunction.

To seek a momentum
which is an eigenproblem for the eigenvalue \( c = \sigma/k \) and the eigenfunction \((\hat{u}, \hat{v}, \hat{\xi})\). Here any perturbation on the forcing due to the feedback from the instability into the incident wave field has been neglected. After discretization by spectral methods, the stability eigenproblem is solved by standard routines [Fauquès and Irandou, 1994] to obtain the dispersion lines \((k, \sigma)\) and the instability curves \((k, \varepsilon)\).


The existence of an instability due to an extremum of the background vorticity at the front side of the velocity profile and the way it coexists with the traditional backshear instability is investigated in this section. In order to have an analytical description a simple case based on BH is considered. An infinitely long flat-bottom beach of constant depth \( h = h_0 \), bounded by a vertical wall at \( x = 0 \), and a piecewise longitudinal current that consists of four regions are assumed (see Figure 1). In region 0, which extends from \( x = 0 \) to \( x = \delta_1 x_0 \) (0 \( \leq \delta_1 \leq 1 \), \( x_0 > 0 \)), the velocity is zero. In region I the current increases linearly from \( V = 0 \) at \( x = \delta_1 x_0 \) to its maximum value \( V = V_0 \) at \( x = \delta_2 x_0 \) (\( \delta_1 < \delta_2 < 1 \)). In region II the velocity varies linearly from \( V_0 \) at \( x = \delta_2 x_0 \) to \( V = 0 \) at \( x = x_0 \). Finally, the velocity current is zero in region III, which extends seaward of \( x = x_0 \).

The corresponding background vorticity is a discontinuous function that is zero in regions 0 and III, has a positive constant value \( V_d/h = f_d/h \) in region I, and has a negative constant value \( V_d/h = f_d/h \) in region II. Notice that for \( \delta_3 = 0 \) the profile coincides with that analyzed by BH, with \( V_d/h \) having only one extremum in region II. For \( \delta_3 > 0 \), there is an additional extremum of the background vorticity in region I.

To seek an analytical solution, bottom friction and turbulent momentum diffusion are disregarded, and the rigid lid assumption is adopted, i.e., \( d\eta/dt \) is negligible in comparison to horizontal fluxes. From this latter hypothesis and the mass conservation (17), a stream function representation of the flow follows:

\[
\begin{align*}
hu(x, y, t) &= -\Psi_y \\
hv(x, y, t) &= \Psi_x
\end{align*}
\]

with

\[
\Psi = \text{Re} \left[ \psi(x)e^{i(\sigma t - \varepsilon y)} \right].
\]

Plugging (19) into (15) and (16) results in an expression for the free surface elevation

\[
\xi(x) = -\frac{1}{gh} \left[ (V - c)\psi_x - V\psi_y \right]
\]

and a governing equation

\[
(V - c) \left[ \psi_{xx} - k^2 \psi = \frac{\psi_h}{h} \right] - h \psi \frac{\partial}{\partial x} \left( \frac{V_x}{h} \right) = 0.
\]

Introducing dimensionless variables \( x', y', \xi', u', v', \Psi' \), and \( t' \) such that

\[
\begin{align*}
x &= x_0 x', \\
y &= x_0 y', \\
\xi &= h_0 \xi', \\
\sigma &= h_0 \sigma', \\
u &= V_0 u', \\
v &= V_0 v', \\
\end{align*}
\]

the simple topography and velocity profile of the model (21) reduces to

\[
\psi_{xx} - k^2 \psi' = 0.
\]

Henceforth the dimensionless variables will be noted without prime.

The boundary conditions \( \psi(x = 0) = 0 \) and \( \psi(x = \infty) = 0 \) are applied to (23) at the shoreline and far offshore. The solution to this boundary problem is

\[
\begin{align*}
\text{Region 0:} & \quad \psi_0 = A_0 \sinh (kx) \\
\text{Region I:} & \quad \psi_1 = A_1 \sinh (kx) + B_1 \cosh (kx) \\
\text{Region II:} & \quad \psi_2 = A_2 \sinh (kx) + B_2 \cosh (kx) \\
\text{Region III:} & \quad \psi_3 = A_3 e^{-kx}
\end{align*}
\]

To ensure the continuity of the stream function \( \psi \) and of the sea surface elevation \( \xi \), the following matching conditions at the interfaces are imposed:

\[
\begin{align*}
\psi_0(\delta_1) &= \psi_1(\delta_1), \\
\psi_1(\delta_2) &= \psi_2(\delta_2), \\
\psi_2(1) &= \psi_3(1), \\
\xi_0(\delta_1) &= \xi_1(\delta_1), \\
\xi_1(\delta_2) &= \xi_2(\delta_2), \\
\xi_2(1) &= \xi_3(1).
\end{align*}
\]

Conditions (25) constitute a linear system for the unknown coefficients \( A_0, A_1, B_1, A_2, B_2, \) and \( A_3 \). A nontrivial solution requires the corresponding matrix to be singular, i.e., a certain condition on \( \sigma \). From (25) the following relations can be obtained:

\[
A_0 = 1 - c_1 B_1 \\
A_1 = \frac{1}{c_1 A_1},
\]

Figure 1. Definition sketch of the extended Bowen and Holman [1989] model.
where
\[ s_1 = \sinh(\delta_2 k), \quad c_1 = \cosh(\delta_2 k), \]
\[ s_2 = \sinh(\delta_2 k), \quad c_2 = \cosh(\delta_2 k), \]
\[ s_0 = \sinh(k), \quad c_0 = \cosh(k), \quad e_0 = e^{-k}. \]

After straightforward algebra the characteristic equation for \( \sigma \) follows from (26) to (29):
\[ \sigma^3 + a_2(k)\sigma^2 + a_1(k)\sigma + a_0(k) = 0, \tag{31} \]
where \( a_2, a_1, \) and \( a_0 \) are real functions of the dimensionless wavenumber \( k \) given by
\[ a_2 = b_0 + b_1 + b_2 + b_3 - b_3 - b_4, \]
\[ a_1 = (b_1 + b_3)(b_2 - b_3) + b_2b_3 + b_0(b_2 + b_3 + b_3) - b_1(b_1 + b_2) - b_1b_3, \tag{32} \]
\[ a_0 = (b_2 - b_4)b_7 + b_0b_3(b_2 + b_3) - b_1b_4b_5 - b_0b_5b_5, \]
where
\[ b_0 = f_s^2, \quad b_1 = -f_s^2c_0, \quad b_2 = -k, \]
\[ b_3 = -(f_t - f_s)sc_0, \quad b_4 = (f_t - f_s)sc_0^2, \tag{33} \]
\[ b_5 = f_sc_0, \quad b_6 = f_sc_0^2, \quad b_7 = (f_t - f_s)c_0^2. \]

For a given \( k \), (31) is a cubic polynomial with real coefficients and has therefore three roots that can be either all real or one real and two complex conjugated, \( \sigma = \sigma_0 \pm i\sigma_1 \), where \( i \) is the imaginary unit and \( \sigma_0 > 0 \) and \( \sigma_1 > 0 \). The basic current is stable for the first case. For the second case the current is unstable, and the positive imaginary part of the complex root, \( \sigma_1 \), is the growth rate of the shear wave.

### 3.1. Analysis of the Frontshear and Backshear Instability Curves

To explain the behavior of the complex solutions of (31), in Figure 2 their imaginary and real parts are presented. Figure 2a shows the instability curves (\( \sigma_0 \) versus \( k \)) calculated with \( \delta_1 = 0.5 \) for different values of \( \delta_2 \). As could be expected, for \( \delta_1 = 0 \) the solution coincides with the BH solution and shows an interval of wave numbers \( k_{n,mn} < k < k_{n,mc} \) (for the example \( k_{n,mn} = 1.38 \) and \( k_{n,mc} = 3.42 \)) for which the shear waves are unstable with a maximum growth rate \( \sigma_{t,0} = 0.33 \) achieved at \( k_0 = 2.5 \).

For small values of \( \delta_1 \) the solution shows two ranges of unstable wave numbers. One of the instability curves has almost the same shape and magnitude as the one obtained by BH, and since it responds to backshear changes, it will be referred to as the backshear instability curve. The additional range \( k_{f,mn} < k < k_{f,mc} \) of unstable wave numbers is clearly associated with the existence of the frontshear \( f_f \) since their growth rates increase with \( f_f \). This curve is referred to as the frontshear instability curve. As an example, for \( \delta_1 = 0.03 \) this interval is \( 0 < k < 1.27 \) and has a maximum growth rate of \( \sigma_{t,f} = 0.05 \), smaller than \( \sigma_{t,b} \), at \( k_f = 1.06 \). For \( \delta_1 = 0.08 \) both curves intersect, showing two relative maxima; again, the one corresponding to the backshear remains almost unchanged. For \( \delta_1 = 0.2 \) there is just one unstable curve with a fastest growth rate \( \sigma_{t,b} = 0.59 \) at \( k = 3.3 \), significantly greater than previous values of \( \sigma_{t,b} \); moreover, the range of unstable wave numbers is wider, extending from \( k_{mn} = 0 \) to \( k_{mc} = 4.92 \). Notice that \( k_{mn} \) is asymptotically zero, which means that when a shear exists at the frontshear region, the range of wavelengths for which the shear waves are unstable is not bounded.

In Figure 2b the values of \( \sigma_0 \) of the unstable modes are highlighted with a thicker curve line. Both the front and backshear instability curves have almost linear dispersion relationships with the same slope.

Figure 3 shows the maximum growth rate \( \sigma_t \) in terms of \( f_f \) for the values \( \delta_2 = 0.2, \delta_2 = 0.5 \), and \( \delta_2 = 0.7 \), which represent three different backslopes. For each backslope \( f_{b,0} \) there is a critical value of the frontshear \( f_f^{c_f} \) (or equivalent, \( f_f^{c_b} \)) above which a critical value \( \delta_1^{c_f} \) or \( \delta_1^{c_b} \) such that for \( f_f < f_f^{c_f} \) the curve has two branches corresponding to the two relative maxima of the instability curves, one due to the frontshear and the other due...
For $f_f > f_f^1$, since the $(k, \sigma_1)$ curve has one maximum, there is just one branch. For $f_f < f_f^1$ the frontshear branch $\sigma_1(f_f)$ increases with $f_f$ starting at $\sigma_1 = 0$, whereas the backshear branch is nearly horizontal, revealing that the existence of the frontshear does not affect the backshear instability.

The behavior of the only branch existing for $f_f > f_f^1$ depends, however, on the relative intensity of the frontshear and backshear. This branch starts at the level of the maximum of both growth rates at $f_f^1$, which for steep ($\delta_2$ large) backslips corresponds to the backshear fastest growing mode (see curves obtained for $\delta_2 = 0.5$ and $0.7$), whereas for mild ($\delta_2$ small) backslips is achieved by the frontshear mode (see curve for $\delta_2 = 0.2$).

Figure 3. Maximum growth rates in terms of $f_f$ for different values of $\delta_2$.

Figure 4. Division of the domain ($\delta_2, \delta_1$) into zones attending to frontshear or backshear wave dominance.
Figure 5a. Dimensionless celerities in terms of $f_f$ for different values of $\delta_2$.

Figure 5b. Dimensionless wave numbers in terms of $f_f$ for different values of $\delta_2$.

Figure 5c. Dimensionless wave frequencies relative to $|f_b|$ in terms of $f_f$ for different values of $\delta_2$. 
For \( f_f > f_f^\delta \) and steep (\( \delta_2 \) large) backslopes there is a range of values for which the backshear instability is dominant since the growth rate is insensitive to the increase of \( f_f \). This pattern is followed up to a second critical value \( f_f^\delta (\delta_2) \) (or equivalently, a critical value \( \delta_2^\delta (\delta_2) \)) for which the growth rate starts to increase with \( f_f \). Here \( f_f^\delta \) establishes the transition from backshear to frontshear predominance for values of \( f_f > f_f^\delta \).

On the contrary, for small values of \( \delta_2 \) that represent mild...
backslips the frontshear instability is dominant at the branching point \( f_f^2 \), having crossed at a critical value \( f_f^2(f_b) < f_f^1 \) the backshear branch. For \( f_f > f_f^2 \) the growth rate increases with increasing \( f_f \) following the tendency of the frontshear branch obtained for values \( f_f < f_f^2 \).

Attending to this behavior, the domain of values \((\delta_2, \delta_1)\) may be divided into four zones that will be referred to as zone A, zone B, zone C, and zone D, delimited by the curves \( \delta_1 = \delta_1^A(\delta_2) \) and \( \delta_1 = \delta_1^A(\delta_2) \) (see Figure 4). In zones A and B the instability curves have two relative maxima; the backshear is dominant in zone A, whereas the frontshear is dominant in zone B. In zones C and D there is only one relative maximum of the growth rate related to the backshear in zone C and to the frontshear in zone D.

### 3.2. Characteristic Timescales and Space Scales

Figure 5 shows, in terms of \( f_f \), the dimensionless velocity phase \( c \) (Figure 5a), the dimensionless wave number \( k \) (Figure 5b), and the dimensionless wave frequency normalized by the absolute value of the backshear, \( \sigma_f/(2\pi f_b) \) (Figure 5c), and by the frontshear, \( \sigma_f/(2\pi f_f) \) (Figure 5d), obtained for \( \delta_2 = 0.2, 0.5, \) and 0.7.

The BH estimate of the shear wave celerity, \( c = V_b/3 \), is still valid no matter whether the instability comes from the frontshear or the backshear. There is an exception: the very long frontshear waves (zones A and B) may have phase speeds significantly smaller increasing with \( f_f \) linearly up to \( c = V_b/5 \).

Moreover, in agreement with BH model the frequency of the backshear mode is proportional to \( f_b, \sigma_f/(2\pi f_b) \approx 0.07 \). On the other hand, the frequency of the frontshear mode tends to be proportional to \( f_f \) also with \( \sigma_f/(2\pi f_f) \approx 0.07 \) for \( \delta_2 = 0.3 \) and \( \delta_2 = 0.5 \); for \( \delta_2 = 0.2 \) a similar trend is observed but with smaller frequencies. Again, for the long frontshear waves (zones A and B) the frequency can be significantly smaller, \( \sigma_f/(2\pi f_f) \approx 0.01 \).

Finally, the wave number of the backshear mode follows the predictions of BH, \( k = \pi \). The wave number of the frontshear wave increases with \( f_f \) ranging from \( k = \pi \) to \( k = 2\pi \) in zone D and up to \( k = 2 \) in zones A and B.

### 3.3. Spatial Structure of the Flow

Once the wave frequency of an unstable shear wave with wave number \( k \) is obtained, fixing the value of \( A_0 \), the coefficients \( A_1, B_1, A_2, B_2, \) and \( A_3 \) can be determined from (26)-(29). The water surface elevation will then be given by (20). In the following, the spatial structure of the shear waves obtained for different pairs of \((\delta_2, \delta_1)\) will be analyzed. Their values are represented in Figure 4 and denoted by \( P_1, P_2, P_3, \) and \( P_4 \).

Figure 6 shows a snapshot of the velocity field of (a) the front shea and (b) the backshear waves with relative faster...
growth rates, calculated for $\delta_2 = 0.8$ and $\delta_1 = 0.03$ (denoted by $P_2$), where the backshear is dominant (zone A). Underneath the arrows the isolines of the stream function have been drawn. Figure 6 also includes the corresponding free surface elevation as a function of $x$. The front shear wave, with a wavelength $k_f = 9.62$, extends from the shoreline over the width of the current. The backshear wave, with a wavelength $k_f = 1.32$, extends over a wider zone of about one and a half the width of the current, and the significant part of the flow due to the backshear is confined to the region $0.5 \leq x \leq 1.5$.

For mild ($\delta_2$ small) backslopes there is a given front slope for which the frontshear and the backshear waves have approximately the same growth rate $\sigma_f$, leading to the possibility of having both instabilities at the same time, which are two propagating waves that have close angular wave frequencies $\sigma_{r,f}$ and $\sigma_{r,p} = \sigma_{r,f} + \Delta \sigma_f$ and close wave numbers $k_f$ and $k_p = k_f + \Delta k$. If the water surface elevation of the frontshear and the backshear waves are $\xi_f(x, y, t) = \Re \{\tilde{\xi}_f(x) \exp[i(k_f y - \sigma_f t)]\}$ and $\xi_p(x, y, t) = \Re \{\tilde{\xi}_p(x) \exp[i(k_p y - \sigma_p t)]\}$, respectively, the resulting wave is then

$$\xi(x, y, t) = \Re \{[\tilde{\xi}_f(x) + \tilde{\xi}_p(x)] e^{i(k_f y - \Delta \sigma_f t)} e^{i(k_p - \sigma_p) t} \},$$

(34)

where $\xi(x, y, t)$ is a wave whose amplitude is modulated in the longitudinal direction with wavelength $\lambda = 2\pi/\Delta k$ and travels in the $y$ direction with celerity $c = \Delta \sigma_f/\Delta k$ (see Figure 7 calculated for $\delta_1 = 0.038$ and $\delta_2 = 0.31$ (denoted by $P_3$ in Figure 4)).

The case analyzed in zone C has the same backshear as the case in zone A, but the frontshear has a larger value. In both cases the backshear is dominant, and the wave field pattern is similar (see Figure 8a obtained for $\delta_1 = 0.2$ and $\delta_2 = 0.8; P_3$ in Figure 4). Finally, in zone D the backshear wave has a smaller wavelength than the one obtained in zone A and extends over a wider zone.

4. Numerical Analysis of Realistic Profiles

The model presented above explains the basic mechanism of instabilities due to an extremum of the background vorticity at the front side of the velocity profile. For a more realistic description the analysis of smooth current profiles in a beach of variable depth has been done by solving (15), (16), and (17) numerically, which takes into account bottom friction and lateral momentum diffusion [see Falqués and Iranzo, 1994]. A series of current profiles with and without background vorticity extremum at the front over a plane sloping beach are analyzed. Then, instability analysis is performed on a current profile obtained from data measured at Leadbetter Beach, and results are compared with field observations.

4.1. Instability Analysis on a Plane Sloping Beach

A series of three longshore currents in a plane beach of slope 1:15, with the same profile seaward of the location of maximum velocity and varying frontshear values, is analyzed.
This moderate reflective profile was chosen expecting an instability behavior comparable to the extended BH model. In all cases the shear extremum at the back is \( V_{x,b} = -0.044 \text{ s}^{-1} \). Case 0 corresponds to a profile without front shear extremum, and cases 1 and 2 have a front shear extremum with values of \( V_{x,f} = 0.085 \text{ s}^{-1} \) and \( V_{x,f} = 0.096 \text{ s}^{-1} \), respectively. Figure 9 shows the three velocity profiles and their respective background vorticities together with the simplified curves for the analysis with the extended BH model.

Figure 10 shows the instability curves obtained for the three cases by neglecting the damping effect of bottom friction and turbulent diffusion (Figure 10a) and with a drag coefficient \( c_d = 0.0001 \) and a kinematic eddy viscosity \( \nu_r \) max = 0.01 m² s⁻¹ (Figure 10b). For the same cases, Figures 11a and 11b show the corresponding dispersion relationships.

Case 0 presents an instability diagram with a single (non-sporious) mode with maximum growth rate achieved at \( k = 0.089 \text{ m}^{-1} \). Cases 1 and 2 present two unstable curves. The fastest growing mode of the first one is placed at \( k = -0.078 \text{ m}^{-1} \) for both cases, and the one of the second curve is placed at \( k = 0.156 \text{ m}^{-1} \) for case 1 and at \( k = 0.201 \text{ m}^{-1} \) for case 2. Similar curves are obtained with and without damping, and the only significant effect of the viscosity and bottom friction is to reduce the growth rates of all modes.

The first modes of cases 1 and 2 resemble the single mode of case 0, and they are insensitive to the change of the front shear value. Their celerities are \( c \sim 0.3 \text{ m s}^{-1} = 3 V_{\text{max}}/4 \), where \( V_{\text{max}} \) is the maximum value of the current, and their wave numbers are in the range \( 2\pi/3 < k l < 2\pi/1.5 \), where \( l = 25 \text{ m} \) is the width of the current. All these values are in the range expected for a vorticity wave due to the extremum of the background vorticity at the backside of the profile. Moreover, the analysis of similar profiles, not presented here for simplicity, have shown that the growth rate increases with \( f_b \).

The second unstable curve exists only for the profiles with a shear at the front. They have similar wave numbers, \( k \sim 0.18 \text{ m}^{-1} \), and their celerities are \( c \sim 0.1 \text{ m s}^{-1} \). The highest growing rate corresponds to the case of the largest front shear.

These results suggest that the first modes of cases 1 and 2 are associated with the extremum of the background vorticity at the back, whereas the second modes are due to the one at the front. Because of the dependence of the background vorticity on the beach profile, numerical results give a front shear wave with a larger wave number than the backshear wave. By applying the analytical model, for the case of two unstable curves the front shear fastest growing mode has a smaller wave number than the corresponding back shear wave. This behavior is due to the differences in shapes of the beach profiles used: a flat sloping bottom was tested for the numerical simulations and a constant water depth for the simplified model. Numerical simulations of the velocity currents of cases 1 and 2 with constant water depth beach profile predicted, as expected.
In order to compare cases 1 and 2 with the extended BH model, simplified current profiles with the same shears at the front and the back sides were analyzed. The parameters used were $V_0 = 0.39 \text{ m s}^{-1}$, $x_0 = 17.28 \text{ m}$, $\delta_1 = 0.48$, and $\delta_2 = 0.22$ for case 1 and $\delta_1 = 0.25$ for case 2, whose corresponding nondimensional shears are $f_0 = -1.92$, $f_1 = 3.84$, and $f_2 = 4.34$ (see Figure 9c). The extended BH model anticipates backshear dominance in both cases with growing rates, $c_{11} = 0.015 \text{ s}^{-1}$ and $c_{12} = 0.017 \text{ s}^{-1}$ (see Figure 10c), which are about twice the values obtained with the numerical model. This may be due to the modelization of the shears in the simplified current profile, to the sloping sea bed, and to the free surface effects [Falqués and Iranzo, 1994]. However, the values of the wave numbers ($k_1 = 0.215 \text{ m}^{-1}$ and $k_2 = 0.249 \text{ m}^{-1}$) and celerities ($c_1 = 0.12 \text{ m s}^{-1}$ and $c_2 = 0.14 \text{ m s}^{-1}$) are in good agreement with the numerical results.

Figure 10 shows the flow structure of (a) the backshear wave of case 0, (b) the backshear wave of case 1, and (c) the front shear wave of case 1. For the velocity profile without a shear extremum at the front the back vorticity wave extends from the shoreline in a zone of width approximately twice the width of the longitudinal current.

The significant part of the front shear wave of case 1 is restricted to a narrow region of width $l = \sim 9 \text{ m}$, delimited approximately by the shoreline and the line of the maximum current velocity, $x = x_m$. Its celerity is roughly $c = 0.5V$, where $V = 0.15 \text{ m s}^{-1}$ is the average velocity in that region, suggesting that the frontshear wave is convected by the longitudinal current between $x = 0$ and $x = x_m$. The presence of the front shear reduces the flow field associated with the backshear wave between the shoreline and the location of maximum current velocity.

4.2. Comparison With Field Data

Theoretical results are compared with field data measured during the 1980 Nearshore Sediment Transport Studies experiment at Leadbetter Beach, Santa Barbara, California. For the analysis the velocity profile fitted by Dodd et al. [1992] to data measured on February 4 (run c) with the model by Thornton and Guza [1986] was chosen as a base. This profile, hereinafter referred to as case $LB_0$, has a single inflexion point located in the zone seaward of the location of maximum velocity, with a value of $V_{max} = -0.024 \text{ s}^{-1}$.

Run c is the third in a series of three runs measured in an ~4 hour interval that were analyzed by Dodd et al. [1992]. To account for variations in tidal elevation and waves over that interval, they fitted three velocity profiles to longshore current measurements, obtaining very similar profiles.

In order to compare the situation analyzed by Dodd et al. [1992] with results for a current profile with an additional shear at the front, a velocity current that conserves the shape of the profile of case $LB_0$ seaward of the location of maximum velocity was adopted. In the shoreward part of the current a cubic spline was fitted to the bin-averaged data from the three consecutive runs. The resulting velocity profile, which will be re-
Figure 12. Snapshot of the flow structure of (a) the backshear wave of case 0, (b) the backshear of case 1, and (c) the frontshear wave of case 1.

ferred to as case \( LB_1 \), has a value of the shear extremum at the front: \( V_{x,f} = 0.033 \text{ s}^{-1} \) (see Figure 13).

For the instability analysis, in order to be consistent with the analysis performed by Dodd et al. [1992], diffusion terms were neglected and the same law for the bottom shear stress was adopted. Different values of the bottom friction coefficient in the range \( 0 \leq c_d \leq 0.009 \) were tested. Case \( LB_0 \) presents an unstable mode with a fastest growing mode at \( k = 0.055 \text{ m}^{-1} \) for values up to \( c_d = 0.007 \) (Figure 14), a value smaller than that used by Dodd et al. [1992] for the calculation of the velocity profile, an inconsistency that was already pointed out by the authors.

The instability frequency-cyclic wave number relationship (Figure 15) agrees with Dodd et al.'s [1992] predictions and falls in the same region as the variance computed from field observations. The instability curves obtained for case \( LB_0 \) with fastest growing modes located at \( k \sim 0.065 \text{ m}^{-1} \), a slightly higher value of \( k \), have larger growth rates and almost

![Figure 13. Velocity profiles of cases \( LB_0 \) and \( LB_1 \) and data measured at Leadbetter beach on February 4.](image-url)
identical dispersion relationships than those obtained for case \( LB_0 \). Moreover, the unstable modes exist for values of \( c_d \) up to 0.009, a value that is the same as that used by Dodd et al. [1992] for the calculation of the velocity profile.

Figure 15 shows (a) the \( f - k \) values measured at Leadbetter on February 4 and the values predicted for (b) case \( LB_0 \) with \( c_d = 0.004 \), (c) case \( LB_1 \) with \( c_d = 0.004 \), and (d) case \( LB \) with \( c_d = 0.009 \). Predicted and measured values agree fairly well.

5. Conclusions

An analytical study and a numerical model [see Falqués and Iranzo, 1994] are used to analyze the instabilities of a longshore current whose background vorticity shows two extrema at both sides of the location of the maximum velocity. The analytical model is based on an idealized triangular velocity profile inspired on BH and defined in terms of two parameters, \( \delta_1 \) and \( \delta_2 \), that determine the intensity of the frontshear and the backshear. A cubic polynomial is obtained as a dispersion relationship, which is solved to obtain the growth rate, the frequency, and the flow pattern of the unstable modes. Because of the limitation imposed by the cubic dispersion relation, only one unstable mode is obtained for each wave number. However, looking at the solution of the response to the backshear and to the frontshear and looking at the shoreward or seaward location of the flow pattern, a backshear wave (BS) or a frontshear wave (FS) can be identified. The dominance of the backshear or the frontshear is discussed in terms of \( \delta_1 \) and \( \delta_2 \); four regions can be distinguished (see Figure 4). In zones A and B the instability curves (growth rate against wave number) show two relative maxima, one can be associated with the BS and another with the FS; in zone A the BS peak is dominant, whereas the FS peak is dominant in zone B. For values of \( (\delta_2, \delta_1) \) in zones C and D the instability curves show only one maximum that can be associated either to the backshear (zone C) or to the frontshear (zone D). In zone B it is possible to find values of \( (\delta_2, \delta_1) \) for which the instabilities associated with the frontshear and the backshear have similar growth rates, with close wave frequencies and wave numbers, leading to the possibility of having a shear wave modulated in the alongshore direction.

For realistic current profiles in a beach of variable depth the numerical analysis confirms the existence of instabilities associated with the frontshear. The frontshear waves extend over the width of the current, and their amplitude is significant only in a region bounded by the shoreline and the location of the maximum current with a characteristic celerity that suggests that the frontshear wave is convected by the portion of the longitudinal current in that region. The presence of the frontshear wave reduces the amplitude of the backshear wave between the shoreline and the location of maximum velocity.

Furthermore, stability analysis was performed numerically on both (1) a profile analyzed by Dodd et al. [1992] with only one extremum of background vorticity and (2) a profile obtained from data measured at Leadbetter Beach that is slightly different at the shoreward region, having an additional extremum of background vorticity. The second profile has a frontshear instability mode that is more unstable than the dominant mode of the first one. In fact, this frontshear mode is still slightly unstable for a bottom shear stress with a drag coefficient of \( c_d = 0.009 \), i.e., the value chosen by Dodd et al. [1992] to achieve a good agreement between the measured and the predicted intensity of the current. Then, the fact that the first
profile is already stable for \( c_d = 0.007 \) leads to the conjecture that the instabilities observed at the Leadbetter experiment could be due to the frontshear instability.

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